Continuous Transformations Preserving All Topological Properties

A Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

1935

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JAMES F. WARDWELL

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CONTINUOUS TRANSFORMATIONS PRESERVING ALL TOPOLOGICAL PROPERTIES.

By JAMES F. WARDWELL.

1. *Introduction.* This paper concerns itself with a solution to the following problem:[1] If A and B are any compact metric spaces and $T(A) = B$ is a continuous transformation, under what conditions will B be homeomorphic with A, that is, what continuous transformations will preserve all topological properties of A? In view of the fact that continuous transformations and upper semi-continuous decompositions[2] are known to be equivalent[3] for compact metric spaces, the above problem can be stated in this way: If we have an upper semi-continuous decomposition of a compact metric space A, under what conditions will the hyperspace B of this decomposition be the same kind of space as A, that is, be homeomorphic with A?

The solution to this problem, for the case when A is a plane or sphere, is due to R. L. Moore.[4] It may be stated as follows: *If A is a topological sphere and $T(A) = B$ is a monotone transformation*[5] *such that, for any $b \in B$, $T^{-1}(b)$ does not separate A, then B is a topological sphere.*

However, no conditions have heretofore been found which yield the desired result for general compact metric spaces or even for any compact Euclidean spaces of higher dimension than two.

2. *Conditions.* A little investigation makes it clear that the conditions which are to be imposed must be conditions on the complements of the sets of the decomposition (that is, the sets $T^{-1}(b)$, for $b \in B$) in the space as well as

[1] This problem was suggested by G. T. Whyburn to whom the author is greatly indebted for his helpful suggestions and criticism in the preparation of this paper. See the abstract of his paper "Analytic topology" in *American Mathematical Monthly*, vol. 42 (1935), p. 190.

[2] See R. L. Moore, *Transactions of the American Mathematical Society*, vol. 27 (1925), pp. 416-428.

[3] See P. Alexandroff, *Mathematische Annalen*, vol. 96 (1927), pp. 551-571, and C. Kuratowski, *Fundamenta Mathematicae*, vol. 11 (1928), pp. 169-185.

[4] See *loc. cit.*

[5] A continuous transformation $T(A) = B$ is said to be a *monotone transformation* when each set $T^{-1}(b)$, for $b \in B$, is connected. See C. B. Morrey, Jr., *American Journal of Mathematics*, vol. 57 (1935), pp. 17-50, and G. T. Whyburn, *American Journal of Mathematics*, vol. 56 (1934), no. 2, pp. 294-302.

on these sets themselves. It seemed that the following condition might yield the desired result:

I. *For any* $b \in B$ *and any* $x \in A$, *there exists a homeomorphism* $W(A - x) = A - T^{-1}(b)$.

However we have not been able to show that, for arbitrary compact metric spaces A and B, B is homeomorphic with A when this condition is satisfied, even when almost all of the sets $T^{-1}(b)$ are degenerate. The same was true for this condition:

II. *For any* $b \in B$ *and any* $x \in T^{-1}(b)$, *there exists a homeomorphism* $W(A - x) = A - T^{-1}(b)$.

This last condition is less restrictive than the first one in that it does not make the space A homogeneous while condition I does. If II is satisfied and if A is homogeneous, then I is obviously satisfied.

We finally found that if condition II was further restricted to give:

III. *For any* $\epsilon > 0$, *any* $b \in B$, *and any* $x \in T^{-1}(b)$, *there exists a homeomorphism* $W(A - x) = A - T^{-1}(b)$ *which is stationary*[6] *outside of the* ϵ*-neighborhood of* $T^{-1}(b)$.

the desired results are obtained for the case stated in section 7 below.

3. *Some effects of these conditions.*

If the compact metric space A *is connected, conditions I and III each imply that* $T(A) = B$ *is a monotone transformation.*

In order to demonstrate this result, let us assume the contrary in both cases. Then there exists some set $T^{-1}(b) = C$, where $C = C_1 + C_2$, mutually separated. Now $\bar{C}_1 \cdot \bar{C}_2 = 0$, since C is closed. Take neighborhoods U_1 and U_2 of C_1 and C_2 respectively so that $\bar{U}_1 \cdot \bar{U}_2 = 0$. Let $U = U_1 + U_2$. There exists a closed cutting S in $A - U$ which separates $\bar{U}_1$ and $\bar{U}_2$ in A, that is, $A - S = A_1 + A_2$, mutually separated, where

$$A_1 \supset \bar{U}_1 \supset C_1 \quad \text{and} \quad A_2 \supset \bar{U}_2 \supset C_2.$$

For condition I, take any $x \in A - S$. Now there exists a homeomorphism $W(A - x) = A - C$. If we apply W^{-1} to the set

$$A - C - S = (A_1 - C_1) + (A_2 - C_2)$$

[6] A transformation W is said to be *stationary* over all points y for which $W(y) = y$.

we obtain a separation

$$A - x - W^{-1}(S) = W^{-1}(A_1 - C_1) + W^{-1}(A_2 - C_2).$$

Now $x + W^{-1}(S)$ is a closed cutting of A, and x is an isolated point of this cutting. Furthermore x is a limit point of both $W^{-1}(A_1 - C_1)$ and $W^{-1}(A_2 - C_2)$. Therefore, by an established theorem,[7] x is a local separating point[7] of A. Now for any point y in $A - x$, there exists a homeomorphism $R(A - x) = A - y$. Hence every point of A is a local separating point of A. Now A is homogenous, and hence every point of A is of the same Menger order. Accordingly no point of A is of order greater than 2 because, by the Local Separating Point–Order Theorem,[7] there exists at most a countable number of local separating points of order greater than 2. Since A consists of more than one point and is connected, it follows that every point of A is of order exactly 2. Therefore A is a simple closed curve. From this it follows that $C_1 + C_2$ separates A; and thus x separates A, which is a contradiction, since no single point separates a simple closed curve. Hence, for condition I, every set $T^{-1}(b)$ is connected, that is, T is a monotone transformation.

For condition III, take any point $x \in C_1$. By hypothesis, there exists a homeomorphism $W(A - x) = A - C$ which is stationary outside of U. Let $R(A) = A$ be the transformation such that, for any $y \in A - C$, $R(y) = W^{-1}(y)$ while $R(C) = x$. Now R is continuous. Let the cutting $S = A - U$. Then we have $A - S = A - (A - U) = U_1 + U_2$, mutually separated. Applying R to $A - S$ we have: $A - R(S) = R(U_1) + R(U_2)$. However, $R(S) = W^{-1}(S) = S$, since W is stationary in $A - U$. Therefore, $A - S = R(U_1) + R(U_2)$. Let P be the set of all points p of U_2 so that $R(p) \subset U_1$. Consequently $R(U_2 - P) \subset U_2$. If we let $Q = S + U_2 - P$, then $A = Q + U_1 + P$. Now it can easily be seen that $Q + U_1$ and P are mutually separated sets. Hence we have a separation of the connected set A; and this is a contradiction. Therefore T is a monotone transformation also in this case.

Since conditions I and III each make the transformation T a monotone one when A is connected, it follows that each of these conditions reduces to those of Moore for the case of the plane.

Condition II does not make T a monotone transformation even when A is connected.

This result is illustrated by the following example. Take a line L in a plane and a segment pq on this line. In the interior of pq take a point p_1,

[7] See G. T. Whyburn, *Monatshefte für Mathematik und Physik*, vol. 36 (1929), pp. 305-314.

and in the interior of p_1q take a point q_1. In pp_1 take a sequence $\{p_i\}$ of points converging to p, ordering these points so that $p_{i+1} \subset pp_i$, for each i. Take a similar sequence $\{q_i\}$ of points in q_1q converging to q. On one side of the line L construct some simple arcs as follows: Take a simple arc C from p_1 to q_1 which meets pq only in the points p_1 and q_1. For each i, take a simple arc C_i from p_i to p_{i+1} which meets the set $pq + C + \sum_{j=1}^{i-1} C_j$ only in p_i and p_{i+1}, and such that $\delta(C_i) \to 0$. For the points of $\{q_i\}$, for each i, take a simple arc D_i from q_i to q_{i+1} which meets $pq + C + \sum_{k=1}^{\infty} C_k + \sum_{j=1}^{i-1} D_j$ only in the points q_i and q_{i+1}, and such that $\delta(D_i) \to 0$. On the other side of L construct sequences of simple closed curves as follows: At each point p_i take a sequence $\{E_k{}^i\}$ of simple closed curves converging to p_i, every two of which intersect only in p_i, and so that $E_k{}^i \cdot E_m{}^j = 0$, for $i \neq j$, and for all k and m. At each point q_i take a sequence $\{F_k{}^i\}$ of simple closed curves converging to q_i, every two of which intersect only in q_i, and so that $F_k{}^i \cdot F_m{}^j = 0$, for $i \neq j$, and for all k and all m, and no one of which intersects any of the $E_k{}^i$, for all i and all k. Let A represent the set

$$pq + C + \sum_{i=1}^{\infty} (C_i + D_i) + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (E_k{}^i + F_k{}^i).$$

Now A is a connected compact metric space. Take a decomposition of A into the set $p_1 + q_1$ and the points of $A - (p_1 + q_1)$. This is obviously an upper semi-continuous decomposition. We must now show that $A - (p_1 + q_1)$ is homeomorphic with $A - p_1$ and with $A - q_1$. Now $A - (p_1 + q_1)$ consists of the sets:

$$(pp_1 + \sum_{i=1}^{\infty} C_i + \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} E_k{}^i) - p_1,$$

$$(q_1q + \sum_{i=1}^{\infty} D_i + \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} F_k{}^i) - q_1,$$

and the countable collection of free open arcs:

$$C - (p_1 + q_1),\ p_1q_1 - (p_1 + q_1),\ E_k{}^1 - p_1, \text{ and } F_k{}^1 - q_1, \text{ for all } k.$$

Furthermore, $A - p_1$ and $A - q_1$ each consist of the same number and same types of sets as $A - (p_1 + q_1)$, and it is easily seen that both $A - p_1$ and $A - q_1$ are homeomorphic with $A - (p_1 + q_1)$.

If A is a 2-dimensional manifold, conditions I and III are equivalent on A.

For, if condition III is given, condition I is satisfied because A is homo-

geneous. If condition I is given, we have, for any set $T^{-1}(b)$, for $b \in B = T(A)$, and any $x \in A$, a homeomorphism $W(A - x) = A - T^{-1}(b)$. Now take the point x in $T^{-1}(b)$. The set $T^{-1}(b)$ is connected since A is connected. Take a monotone sequence $\{U_i\}$ of neighborhoods closing down on x such that $\bar{U}_i$ is a closed 2-cell, for each i. Let $W(U_i - x) = V_i^\circ$ and $V_i^\circ + T^{-1}(b) = V_i$, for each i. The neighborhoods V_i close down on $T^{-1}(b)$ since the neighborhoods U_i close down on x. Let $C_i = U_i - \bar{U}_{i+1}$ and $D_i = V_i - \bar{V}_{i+1}$, for each i. Now $W[F(U_i)] = F(V_i)$,[8] for each i, and W is a homeomorphism. Hence for each i, $C_i + F(U_i) + F(U_{i+1})$ and $D_i + F(V_i) + F(V_{i+1})$ are each homeomorphic with a circular ring. For any $\epsilon > 0$ take an ϵ-neighborhood U_ϵ of $T^{-1}(b)$ so that $U_\epsilon - T^{-1}(b)$ is homeomorphic with a 2-cell V_ϵ of x minus x. Pick a k large enough so that $\bar{V}_k \subset U_\epsilon$ and $\bar{U}_k \subset U_\epsilon$. Now there exists a homeomorphism

$$R_\epsilon[(U_\epsilon - \bar{U}_k) + F(U_k) + F(U_\epsilon)] = (U_\epsilon - \bar{V}_k) + F(V_k) + F(U_\epsilon),$$

where $R_\epsilon(p) = p$ for $p \in F(U_\epsilon)$, since each of these sets is a circular ring and clearly any homeomorphism between the two outer curves of two such rings can be extended to the whole rings.[9] Let $R_\epsilon = S_{k-1}$. Similarly there exists a homeomorphism

$$S_k[C_k + F(U_k) + F(U_{k+1})] = D_k + F(V_k) + F(V_{k+1})$$

such that $S_k(x) = S_{k-1}(x)$ for $x \in F(U_k)$. Likewise there exists a homeomorphism

$$S_{k+1}[C_{k+1} + F(U_{k+1}) + F(U_{k+2})] = D_{k+1} + F(V_{k+1}) + F(V_{k+2})$$

such that $S_{k+1}(x) = S_k(x)$ for $x \in F(U_{k+1})$, and so on. In general for $j \geq -1$ we have a homeomorphism

$$S_{k+j}[C_{k+j} + F(U_{k+j}) + F(U_{k+j+1})] = D_{k+j} + F(V_{k+j}) + F(V_{k+j+1})$$

such that $S_{k+j}(x) = S_{k+j-1}(x)$ for $x \in F(U_{k+j})$.

Now define a transformation S as follows: $S(p) = p$ if $p \in M - U_\epsilon$, $S(p) = R_\epsilon(p)$ if $p \in (U_\epsilon - \bar{U}_k) + F(U_k) + F(U_\epsilon)$, $S(p) = S_j(p)$ for $j \geq k$ where j is the least integer such that p is a point of $C_j + F(U_j) + F(U_{j+1})$. Now $S(A - x) = A - T^{-1}(b)$ is a homeomorphism which is stationary outside of U_ϵ. Hence condition III is satisfied.

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[8] The boundary $\bar{P} - P$ of any open set P is represented by $F(P)$.

[9] See Schoenflies, *Mathematische Annalen*, vol. 62 (1906), p. 324.

4. *Notation.* If A and B are any compact metric spaces and $T(A) = B$ is a continuous transformation, let $G = G_0$ denote the collection of all non-degenerate sets $T^{-1}(b)$, for $b \in B$ in A. Let G_1 denote the collection of all sets of G which intersect $L = L_0 =$ lim. sup. G. Let G_2 be the collection of all sets of G_1 which intersect $L_1 =$ lim. sup. G_1, etc.

If the collection G is countable, let $\{g_i^k\}$ represent the sets of $G_{k-1} - G_k$, for $k = 1, 2, \cdots$. Cover each g_i^k by a neighborhood U_i^k so that: (a) $U_i^k \cdot g_j^k = 0$, for $j \neq i$; (b) $U_i^k \cdot \sum_{j-1}^{i-1} \bar{U}_j^k = 0$; (c) no set of G_k intersects U_i^k, for any i; and (d) $\bar{U}_i^k$ is contained in the $\rho(g_i^k, L_{k-1})/2$-neighborhood of g_i^k, for each i.

5. Lemma. *If A and B are compact metric spaces and $T(A) = B$ is continuous and such that for any $\epsilon > 0$, any $b \in B$, and any $x \in T^{-1}(b)$, there exists a homeomorphism $W(A - x) = A - T^{-1}(b)$ which is stationary outside of the ϵ-neighborhood of $T^{-1}(b)$, and if there exists an integer k so that $G_k = 0$, then A is homeomorphic with B.*

Proof. 1°. If $G_i = 0$, for any integer i, then G_{i-1} is a null sequence.[10] For, if not, there exists a number $d > 0$ such that there is an infinite collection $\{h_i\}$ of sets of G_{i-1} the diameter $\delta(h_i)$ of each of which is greater than d. Hence, if we take a convergent subsequence $\{h'_i\}$ of $\{h_i\}$, $\delta(\text{lim. sup. } \{h'_i\}) \geq d$. Now lim. sup. $\{h'_i\} \subset T^{-1}(p)$ for some $p \in B$. Therefore $\delta(T^{-1}(p)) \geq d$, and thus $T^{-1}(p) \in G_i$, since L_{i-1} contains lim. sup. $\{h_i\}$. This contradicts the fact that $G_i = 0$.

We shall make the proof by induction.

2°. We shall first demonstrate the result for the case $k = 1$. From 1° it follows that G is a null sequence, since $G_1 = 0$. Now $\{g_i^1\}$ denotes the collection of sets of G. Take a point $y_i \in g_i^1$, for each i. Let $T(g_i^1) = b_i$, for each i. By hypothesis, there exists a homeomorphism [11] $W_1(A - y_1) = A - g_1^1$ which is stationary in $A - U_1^1$. Now $T W_1(A - y_1) = B - b_1$. Let $T_1(A) = B$ be the transformation such that for any $x \in A$, $T_1(x) = T W_1(x)$ when $x \in A - y_1$, and $T_1(y_1) = b_1$. Now T_1 is univalued and continuous, since T and W_1 are, and since A is compact. Now T_1 is such that $T_1^{-1}(b_i) = g_i^1$, for $i = 2, \cdots$, since $T_1 = T$ over $A - U_1^1$. Similarly, for

[10] A sequence $\{M_i\}$ of sets is said to be a *null sequence* provided that, for any $\epsilon > 0$, there are at most a finite number of the sets which have a diameter greater than ϵ.

[11] The method used here was suggested by a proof given by Mrs. Lucille Whyburn for a certain finite case of our problem.

each n, there exists a homeomorphism $W_n(A - y_n) = A - g_n^1$ which is stationary in $A - U_n^1$. Moreover $g_n^1 \equiv T_{n-1}^{-1}(b_n)$, and thus

$$W_n(A - y_n) = A - T_{n-1}^{-1}(b_n).$$

Hence $T_{n-1}W_n(A - y_n) = B - b_n$. Let $T_n(A) = B$ be the transformation such that, for $x \in A - y_n$, $T_n(x) = T_{n-1}W_n(x)$, and $T_n(y_n) = b_n$. Now T_n is also univalued and continuous. Furthermore $T_n^{-1}(b_i) = g_i^1$, for $i > n$, and $T_n^{-1}(b_j)$ is a single point for each $j \leq n$. From our definition of $T_n(A) = B$ it follows that, for any n, and any $x \in A$:

$$\begin{aligned}
&T_n(x) = T_{n-1}W_n(x) && \text{except for } x = y_n,\\
&T_n(x) = T_{n-2}W_{n-1}W_n(x) && \text{except for } x = y_n \text{ or } y_{n-1},\\
&\cdot \quad \cdot && \cdot \quad \cdot \quad \cdot\\
&\cdot \quad \cdot && \cdot \quad \cdot \quad \cdot\\
&\cdot \quad \cdot && \cdot \quad \cdot \quad \cdot\\
&\cdot \quad \cdot && \cdot \quad \cdot \quad \cdot\\
&T_n(x) = T_1W_2 \cdot \cdot \cdot W_n(x) && \text{except for } x \in \sum_{i=2}^{n} y_i,\\
&T_n(x) = T\,W_1W_2 \cdot \cdot \cdot W_n(x) && \text{except for } x \in \sum_{i=1}^{n} y_i.
\end{aligned}$$

Let $S = \lim_{n=\infty} \{T_n\}$. We will now prove that $\{T_n\}$ is uniformly convergent. Since all T_n are continuous over A, it will follow that S is also continuous. Since T is continuous and is defined over a compact space, it is uniformly continuous. Hence, for any $\epsilon > 0$, there exists a δ_ϵ such that, if $\rho(x, y) < \delta_\epsilon$, for $x, y \in A$, $\rho(T(x), T(y)) < \epsilon$. Take any $\epsilon > 0$. Let $\delta = \delta_{\epsilon/2}$. Cover each point of L by a $\delta/2$-neighborhood $V_{\delta/2}$ and by a δ-neighborhood V_δ. Since L is closed and compact we can find a finite number of the neighborhoods $V_{\delta/2}$, say $V^1{}_{\delta/2}, V^2{}_{\delta/2}, \cdots V^h{}_{\delta/2}$, whose sum covers L. Let $V_\delta{}^1, V_\delta{}^2, \cdots V_\delta{}^h$, be the corresponding δ-neighborhoods. Now $V = \sum_{i=1}^{h} V_\delta{}^i \supset L$. Since $\{g_i{}^1\}$ is a null sequence and because of our definition of the neighborhoods $U_i{}^1$, it follows that $\{U_i{}^1\}$ s a null sequence and lim. sup. $\{U_i{}^1\} = L$. Hence there exists an integer N such that $\delta(U_n{}^1) < \delta/2$, for all $n > N$. There also exists an integer M so that $U_m \cdot \sum_{i=1}^{h} V^i{}_{\delta/2} \neq 0$, for all $m > M$. For, if not, there would be an infinite collection of the sets of $\{U_i{}^1\}$ which did not intersect $\sum_{i=1}^{h} V^i{}_{\delta/2}$. Since A is compact, this collection would possess a limit point p which is contained in L but not contained in $\sum_{i=1}^{h} V^i{}_{\delta/2}$, which is a contradiction.

Let N_ϵ be the larger of the two numbers N and M. For any $n > N_\epsilon$, U_n^1 is contained in some V_δ^r, for $r = 1, 2, \cdots$, or h. For $U_n^1 \cdot V_{\delta/2}^r \neq 0$, for some r, since $n > M$. Also $\delta(U_n^1) < \delta/2$ since $n > N$. Hence $U_n^1 \subset V_\delta^r$.

For any point $x \in (A - \sum_{i=N_\epsilon+1}^{\infty} U_i^1)$, $T_n(x) = T_{N_\epsilon} W_{N_\epsilon+1} \cdots W_n(x)$ and $T_{n+p}(x) = T_{N_\epsilon} W_{N_\epsilon+1} \cdots W_{n+p}(x)$, for $n > N_\epsilon$. Now all W_j, for $j > N_\epsilon$, are stationary outside of $\sum_{i=N_\epsilon+1}^{\infty} U_i^1$. Hence $T_n(x) = T_{N_\epsilon}(x) = T_{n+p}(x)$. Therefore, for any such x, $\rho(T_n(x), T_{n+p}(x)) = 0$, for $n > N_\epsilon$, and $p = 1, 2, \cdots$.

For any $x \in \sum_{i=N_\epsilon+1}^{\infty} (U_i^1 - y_i)$, we have that $T_n(x) = T\, W_{N_\epsilon+1} \cdots W_n(x)$, since, by our definition of the neighborhoods U_i^1, all W_j, for $j \leq N_\epsilon$ are stationary for all such points x. Now x is contained in U_h^1, for some h; and U_h^1 is contained in V_δ^r, for some r. Each W_k, for $N_\epsilon < k \leq n$, transforms a point of U_h^1 only into some other point of U_h^1. Therefore $y = W_{N_\epsilon+1} \cdots W_n(x)$ is contained in U_h^1 which is contained in V_δ^r. Likewise

$$z = W_{N_\epsilon+1} \cdots W_{n+p}(x) \subset U_h^1 \subset V_\delta^r.$$

Hence

$$\rho(y, z) < \delta_{\epsilon/2}.$$

However

$$T_n(x) = T\, W_{N_\epsilon+1} \cdots W_n(x) = T(y)$$

and

$$T_{n+p}(x) = T\, W_{N_\epsilon+1} \cdots W_{n+p}(x) = T(z).$$

Accordingly

$$\rho(T_n(x), T_{n+p}(x)) = \rho(T(y), T(z)) < \epsilon, \text{ for } n > N_\epsilon, \text{ and } p = 1, 2, \cdots,$$

since T is uniformly continuous. For any y_i, $T_i(y_i) = b_i$, by definition. But $b_i = T(y_i)$. Hence $T_i(y_i) = T(y_i)$, for all i. If $x = y_i$, for any $i > N_\epsilon$, $T_n(x) = T_i W_{i+1} \cdots W_n(x)$, for $i > N_\epsilon$, $n > N_\epsilon$, and $n > i$. But all W_j, for $i < j \leq n$, are stationary on y_i. Therefore

$$T_n(x) = T_i(x) = T(x), \text{ for all } i < n.$$

If $n \leq i$, $T_n(x) = T(x)$, by definition. In the same way $T_{n+p}(x) = T(x)$, for $i > N_\epsilon$, $n > N_\epsilon$. Hence

$$\rho(T_n(x), T_{n+p}(x)) = 0, \text{ for } n > N_\epsilon,\ p = 1, 2, \cdots, \text{ for } x \subset \sum_{i=N_\epsilon+1}^{\infty} y_i.$$

Therefore, for any $\epsilon > 0$, there exists an N_ϵ such that for any $n > N_\epsilon$, and for any

$$x \in A,\ \rho(T_n(x),\ T_{n+p}(x)) < \epsilon, \text{ for } p = 1, 2, \cdots.$$

Hence $\{T_n\}$ converges uniformly to S. Consequently S is continuous.

Furthermore S is univalued both ways. For take any two distinct points p and q of A. There exists an integer N such that $p+q \subset A - \sum_{i=N}^{\infty} U_i^1$ Moreover T_N is univalued both ways over $A - \sum_{i=N}^{\infty} U_i^1$. Therefore

$$T_N(p) \neq T_N(q).$$

Now for any point

$$x \in A - \sum_{i=N}^{\infty} U_i^1, \ T_{N+j}(x) \equiv T_N(x), \text{ for all } j.$$

Hence

$$S(p) = T_{N+j}(p) = T_N(p) \neq T_N(q) = T_{N+j}(q) = S(q).$$

Accordingly $S(A) = B$ is a homeomorphism.

3°. We now assume that the lemma is true when $G_{k-1} = 0$ and proceed to prove that it is then true when $G_{k-1} \neq 0$ but $G_k = 0$.

Take a decomposition of A into the sets of G_{k-1} and the points of $A - G_{k-1}$. This is obviously an upper semi-continuous decomposition. Let its hyperspace be C and let $t(A) = C$ be the continuous transformation associated with the decomposition. The sets of G_{k-1} form a null sequence $\{g_i^k\}$ since $G_k = 0$. Let $t(g_i^k) = c_i \in C$, for each i. Now for any point $c \in C$, $t^{-1}(c)$ is a single point unless $c = c_i$, for some i; and $t^{-1}(c_i) = g_i^k = T^{-1}(b)$, a non-degenerate set, for some $b \in B$. Hence $\{t^{-1}(c_i)\}$ is the null sequence $\{g_i^k\}$. Furthermore, lim. sup. $\{t^{-1}(c_i)\}$ = lim. sup. $\{g_i^k\}$ which is L_{k-1}, and the collection of sets of G which intersect L_{k-1} is vacuous, by hypothesis. Moreover, for any $\epsilon > 0$, any $t^{-1}(c)$, for $c \in C$, and any $y \in t^{-1}(c)$, there exists a homeomorphism $W(A - y) = A - t^{-1}(c)$ which is stationary outside of the ϵ-neighborhood of $t^{-1}(c)$, because any non-degenerate $t^{-1}(c)$ is a set $T^{-1}(b)$, for some $b \in B$. Hence the conditions of this lemma are satisfied by $t(A) = C$ for the case demonstrated in part 2°. Therefore A is homeomorphic with C.

Let $Z(C) = T\,t^{-1}(C) = B$. Then $Z(C) = B$ is a univalued transformation. Moreover it is continuous, since T and t are each continuous, and A is compact. All $Z^{-1}(b)$, for $b \in B$, are degenerate except when $T^{-1}(b)$ is a set of $G - G_{k-1}$. Let H be the collection of all non-degenerate sets $Z^{-1}(b)$, for $b \in B$. Let H_1 be the collection of all sets of H which intersect M = lim. sup. H. Let H_2 be the collection of all sets of H_1 which intersect M_1 = lim. sup. H_1, etc. Wherefore, $H = t(G)$, and $H_i = t(G_i)$, for each $i < k$. However $t(G_{k-1})$ consists of the collection $\{c_i\}$ of single points of C. Hence $H_{k-1} = 0$.

Take any $\epsilon > 0$, any non-degenerate $Z^{-1}(b)$, for $b \in B$, and any point $y \in Z^{-1}(b)$. We must show that there exists a homeomorphism

$$W(C - y) = C - Z^{-1}(b)$$

which is stationary outside of the ϵ-neighborhood U_ϵ of $Z^{-1}(b)$ in C.

Now t is uniformly continuous, since A is compact. Hence there exists a δ_ϵ so that if $\rho(r, s) < \delta_\epsilon$, for $r, s \in A$, then $\rho(t(r), t(s)) < \epsilon$. Take a neighborhood V of $T^{-1}(b)$ in A such that V is contained in the δ_ϵ-neighborhood of $T^{-1}(b)$, and such that $V \cdot g_i^k = 0$, for all i. We can do this because the sets $T^{-1}(p)$, for $p \in B$, are closed and disjoint, and $T^{-1}(b) \cdot \bar{G}_{k-1} = 0$, since $Z^{-1}(b)$ is non-degenerate in C. If we let $x = t^{-1}(y)$, then $x \in T^{-1}(b)$. Now, by hypothesis, there exists a homeomorphism $R(A - x) = A - T^{-1}(b)$ which is stationary in $A - V$. However, $A - x = t^{-1}(C - y)$. Hence

$$R\, t^{-1}(C - y) = A - T^{-1}(b).$$

We now designate by W the transformation

$$t\, R\, t^{-1}(C - y) = C - Z^{-1}(b).$$

Now $W(C - y) = C - Z^{-1}(b)$ is univalued. For take any point $q \in C - y$. If $q \neq c_i$, for any i, $W(q) = t\, R\, t^{-1}(q)$ is a single point of $C - Z^{-1}(b)$, because t is one-to-one for such points q and R is a homeomorphism. If $q = c_i$, for some i, then $t^{-1}(q) = g_i^k$. Furthermore $R\, t^{-1}(q) = t^{-1}(q)$, since $g_i^k \cdot V = 0$ and R is stationary outside of V. Therefore $W(q) = t\, t^{-1}(q) = q$. By a similar proof we see that W^{-1} is also univalued. Hence W is one-to-one.

Furthermore, W and W^{-1} are both continuous since t and R are continuous and A is compact. Therefore W is a homeomorphism.

Now W is stationary outside of U_ϵ. For if we take any point $q \in C - U_\epsilon$, then $t^{-1}(q) \cdot V = 0$. Hence, since R is stationary outside of V,

$$R\, t^{-1}(q) = t^{-1}(q).$$

Thus we have

$$W(q) = t\, R\, t^{-1}(q) = t\, t^{-1}(q) = q.$$

Accordingly the homeomorphism

$$W(C - y) = C - Z^{-1}(b)$$

is stationary outside of U_ϵ.

Hence the conditions of the lemma are satisfied by $Z(C) = B$ for the case $k - 1$. We have assumed that for this case the lemma is true. Wherefore, C is homeomorphic with B. However, we have seen that A is homeomorphic with C. Therefore A is homeomorphic with B.

6. *Corollaries.* There are some properties of the homeomorphism, say $S(A) = B$, whose existence is established in the above lemma, which follow from the proof of the lemma. These will be stated and verified in the following two corollaries. We will assume that the conditions of the lemma are satisfied.

COROLLARY I. *The homeomorphism $S(A) = B$, whose existence is established in the lemma, is such that $S \equiv T$ over all points of A lying in $A - \sum_{j=1}^{k} \sum_{i=1}^{\infty} U_i^j$.*

Proof. We shall make the proof by induction. Take any point p in $A - \sum_{j=1}^{k} \sum_{i=1}^{\infty} U_i^j$. If $k = 1$, then $\{g_i^1\}$ is a null sequence. Now $W_j(p) = p$, for all j, since W_j is stationary outside of U_j^1, for every j. Now, for all m, $T_m(p) = T\,W_1 \cdots W_m(p)$. Hence $T_m(p) = T(p)$, for all m. Therefore $S(p) \equiv T(p)$ for this case.

Now let us assume that this is also true for $k - 1$, and prove that it is then true for k. The sequence $\{g_i^k\}$ is a null sequence, since $G_k = 0$. Take a decomposition of A into the sets of $\{g_i^k\}$ and the points of $A - \sum_{i=1}^{\infty} g_i^k$. This is an upper semi-continuous decomposition. Let its hyper-space be C and let $t(A) = C$ be the continuous transformation associated with the decomposition. Now $t(A) = C$ satisfies the conditions of the lemma for the case $k = 1$. Hence there exists a homeomorphism $R_1(A) = C$ so that $R_1 \equiv t$ over all points of the set $A - \sum_{i=1}^{\infty} U_i^k$, as we have just demonstrated above. Now we established in part 3° of the proof of the lemma that the conditions of the lemma are satisfied for the case $k - 1$ by the spaces C and B and the transformation $Z(C) = T\,t^{-1}(C) = B$, where the non-degenerate sets in C are the sets $t(g_i^j)$ and the corresponding neighborhoods are the neighborhoods $t(U_i^j)$ for all i, and for $j < k$. From our assumption it then follows that there exists a homeomorphism $R_2(C) = B$ such that $R_2 \equiv Z$ over $C - \sum_{j=1}^{k-1} \sum_{i=1}^{\infty} t(U_i^j)$. Now $S(A) = R_2R_1(A) = B$. Furthermore $S(p) = R_2R_1(p) = R_2t(p)$, since $R_1(p) \equiv t(p)$. Now $t(p) \subset C - \sum_{j=1}^{k-1} \sum_{i=1}^{\infty} t(U_i^j)$. Hence $R_2t(p) = Z\,t(p)$. However $Z\,t(p) = T\,t^{-1}t(p) = T(p)$. Thus $S(p) = T(p)$ for any point p in $A - \sum_{j=1}^{k} \sum_{i=1}^{\infty} U_i^j$.

COROLLARY II. *The homeomorphism $S(A) = B$, whose existence is established in the lemma, is such that, for any $p \in A$, if $p \subset U_a^a$, for any*

$\alpha \leq k$, and any d, $S^{-1}T(p) \subset U_r^{\rho}$, *for some* $\rho \leq k$ *and some* r, *such that there is a chain of neighborhoods* $U_d^{\alpha}, U_e^{\beta}, \cdots, U_r^{\rho}$ *so that each two consecutive neighborhoods in this chain intersect each other.*

Proof. We shall make the proof by induction.

If $k = 1$, then $\alpha = 1$. Now, for any j, $S(U_j^1) = T_j(U_j^1)$, since, for any point $x \in U_j^1$, $T_n(x) = T_j(x)$ for all $n > j$. Furthermore

$$T_j(U_j^1 - y_j) = T\,W_j(U_j^1 - y_j)$$

by definition, and because all W_i are stationary over U_j^1, for $i < j$, since $U_j^1 \cdot U_i^1 = 0$ for $i \neq j$. Now, since W_j is stationary in $A - U_j^1$, we have

$$W_j(U_j^1 - y_j) = U_j^1 - g_j^1.$$

Hence we have

$$S(U_j^1) - S(y_j) = S(U_j^1 - y_j) = T(U_j^1 - g_j^1) = T(U_j^1) - T(g_j^1).$$

But $S(y_j) = b_j = T(g_j^1)$. Therefore $S(U_j^1) = T(U_j^1)$. Thus, for any point $p \subset U_j^1$, for any j, $S^{-1}T(p) \subset U_j^1$.

We now assume that the statement is true for $k - 1$ and proceed to prove it for k. The sequence $G_{k-1} = \{g_i^k\}$ is a null sequence, since $G_k = 0$. Take a decomposition of A into the sets of G_{k-1} and the points of $A - G_{k-1}$, and thus obtain the hyperspace C, the transformations

$$t(A) = C,\ Z(C) = T\,t^{-1}(C) = B$$

and the homeomorphisms

$$R_1(A) = C,\ R_2(C) = B \text{ and } S(A) = R_2R_1(A) = B$$

as in the proof of Corollary I. From our assumption it follows that $R_2(C) = B$ is such that $R_2^{-1}Z(C) = C$ satisfies the results of this corollary for the case $k - 1$. Now $T(A) = Z\,t(A) = B$. Hence we have that

$$S^{-1}T(A) = R_1^{-1}R_2^{-1}Z\,t(A) = A.$$

Take any point p of A. If $p \subset U_i^j$, for any $j \leq k$, and any i, then

$$q = t(p) \subset t(U_i^j).$$

According to our assumption, if $q \subset t(U_d^{\alpha})$, for any $\alpha \leq k - 1$, and any d, $R_2^{-1}Z(q)$ is in a neighborhood $t(U_r^{\rho})$ for some ρ and some r, such that there is a chain of neighborhoods $t(U_d^{\alpha}), t(U_e^{\beta}), \cdots, t(U_r^{\rho})$ such that each two consecutive neighborhoods in this chain intersect each other. Now if

$$R_2^{-1}Z(q) \not\subset \sum_{i=1}^{\infty} t(U_i^k),$$

then

$$S^{-1}T(p) = R_1^{-1}[R_2^{-1}Z(q)] \subset U_r,$$

since

$$R_1(x) = t(x) \text{ for any } x \in A - \sum_{i=1}^{\infty} U_i^k.$$

Now the chain of neighborhoods $U_i^j, U_d^a, \cdots, U_r^\rho$ is such that each two consecutive neighborhoods of the chain intersect each other. For $U_i^j \cdot U_d^a \neq 0$ since $t(U_i^j) \cdot t(U_d^a) \supset q$; and any other two consecutive ones intersect because their images under t do. Hence the corollary is established for this case. Now if $R_2^{-1}Z(q) \subset t(U_m^k)$, for some m, then we have that

$$S^{-1}T(p) = R_1^{-1}[R_2^{-1}Z(q)] \subset U_m^k,$$

since, for all m, $R_1(U_m^k) = t(U_m^k)$, by the first part of this proof. The chain $U_i^j, U_d^a, \cdots, U_r^\rho, U_m^k$ is obviously such that each two consecutive neighborhoods of the chain intersect. For

$$U_r^\rho \cdot U_m^k \neq 0 \text{ since } t(U_r^\rho) \cdot t(U_m^k) \supset R_2^{-1}Z(q),$$

and by the above argument. Hence the corollary is also established for this case.

7. THEOREM. *If A and B are compact metric spaces and $T(A) = B$ is continuous and satisfies the conditions:* (1) *for any $\epsilon > 0$, any set $T^{-1}(b)$, for $b \in B$, and any $x \in T^{-1}(b)$, there exists a homeomorphism*

$$W(A - x) = A - T^{-1}(b)$$

which is stationary outside of the ϵ-neighborhood of $T^{-1}(b)$; (2) *there exists some number α of the first or second number class such that $G_\alpha = 0$; and* (3) $\prod_{0}^{\infty} L_i$ *is a zero-dimensional set, then A is homeomorphic with B.*

Proof. The theorem is true for any finite number α, by the lemma. We will now prove that the theorem is true when $\alpha = \omega$. When this is established, it will follow that it is true for any $\alpha > \omega$. For, if α is an isolated number, the result is obtained by induction as in the lemma. If α is a limit number, we again use induction, that is, we suppose the theorem true for all $\beta < \alpha$, and then prove that it is also true for α by the same method of proof we shall use here for $\alpha = \omega$.

Let H_1 be the collection of all sets g_j^1 of $G - G_1$ such that $U_j^1 \cdot \sum_{r=2}^{\infty} \sum_{i=1}^{\infty} U_i^r = 0$. Let $\{h_i^1\}$ represent the collection H_1. Let V_i^1 denote the neighborhood U_k^1 which covers h_i^1, for each i. Let H_2 be the collection of all sets of $(G - H_2) - G_2$ whose neighborhoods U do not intersect $\sum_{r=3}^{\infty} \sum_{i=1}^{\infty} U_i^r$. Let $\{h_i^2\}$ represent the collection H_2; and let V_i^2 denote the neighborhood U_k^s, $s = 1$ or 2, which covers h_i^2, for each i. Let H_3 be the collection of all sets of $(G - \sum_{k=1}^{2} H_k) - G_3$ whose neighborhoods U do not intersect $\sum_{r=4}^{\infty} \sum_{i=1}^{\infty} U_i^r$. Let $\{h_i^3\}$ denote the collection H_3; and let V_i^3 denote the neighborhood U_k^s. $s = 1$, 2, or 3, which covers h_i^3, for each i. Continue in this way indefinitely.

Now $\sum_{i=1}^{\infty} H_i = G$. For take any $g \in G$. There exists a finite integer N which is the largest number so that $g \in G_N$. Now, by definition, the neighborhood U of g is contained in the $\rho(g, L_N)/2$-neighborhood of g, thus $\bar{U} \cdot L_N = 0$. Take a neighborhood V of L_N so that $V \cdot \bar{U} = 0$. Now there exists an integer M so that $U_i^m \subset V$, for all $m > M$ and all i, since lim. sup. $\sum_{j=N+1}^{\infty} \sum_{i=1}^{\infty} U_i^j = L_N$. Therefore U does not intersect $\sum_{r=M+1}^{\infty} \sum_{i=1}^{\infty} U_i^r$. Hence $g \in H_s$ for some $s \leq M$.

Take a decomposition of A into the sets of H_1 and the points of $A - H_1$. This is obviously an upper semi-continuous decomposition. Let its hyperspace be C_1 and let $T_1(A) = C_1$ be the continuous transformation associated with the decomposition. Now no sets of H_1 intersect lim. sup. H_1. Therefore $T_1(A) = C_1$ satisfies the conditions of the lemma for the case $k = 1$. Hence, by Corollary I section 6, there exists a homeomorphism $S_1(A) = C_1$ such that $S_1 \equiv T_1$ over $A - \sum_{i=1}^{\infty} V_i^1$. Let $Z_1(A) = S_1^{-1}T_1(A) = A$. Now Z_1 is univalued and continuous; $Z_1(h_i^1)$ is a point of A, for each i; and $Z_1(h_i^k) = h_i^k$, for all $k > 1$ and for all i.

Take a decomposition of A into the sets of H_2 and the points of $A - H_2$. This is an upper semi-continuous decomposition. Let C_2 be its hyperspace and $T_2(A) = C_2$ its associated continuous transformation. Now $T_2(A) = C_2$ satisfies the conditions of the lemma for the case $k = 2$. Hence there exists a homeomorphism $S_2(A) = C_2$ such that $S_2 \equiv T_2$ over $A - \sum_{i=1}^{\infty} V_i^2$. Let $Z_2(A) = S_2^{-1}T_2Z_1(A) = A$. Now Z_2 is univalued and continuous and such that: $Z_2(h_i^j)$ is a point of A, for $j = 1$ or 2 and for each i; and Z_2 is the identity transformation over $A - \sum_{k=1}^{2} \sum_{i=1}^{\infty} V_i^k$. Hence $Z_2(h_i^k) = h_i^k$, for all

$k > 2$ and for all i. Continuing in this way indefinitely we get a sequence $\{Z_n\}$ of such transformations.

Let $Z = \lim_{n=\infty}. \{Z_n\}$. We shall now prove that $\{Z_n\}$ converges uniformly to Z. It will then follow that Z is continuous. Take any $\epsilon > 0$. Now since $\prod_0^\infty L_i$ is closed, compact, and zero-dimensional, we can cover it by a finite number of disjoint $\epsilon/2$-neighborhoods, say $W_1, W_2, \cdots, W_h$. Let W denote $\sum_{i=1}^{h} W_i$. Now there exists an M such that $\sum_{j=M+1}^{\infty} \sum_{i=1}^{\infty} V_i^j \subset W$. For, if not, then no matter what M we take, there exists some $m > M$ and some i so that $V_i^m \not\subset W$. Take a sequence $M_1 < M_2 < M_3 < \cdots$ of integers and take the corresponding $V_i^{m_j} \not\subset W$. Take a point p_{m_j} of $V_i^{m_j}$ but not contained in W. Now $\{p_{m_j}\}$ contains a convergent subsequence $\{p'_{m_j}\}$ which converges to a point p. Now $p \subset L$ because lim. sup. $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} V_i^j = L$. Also, for any finite integer n, infinitely many of the points of $\{p'_{m_j}\}$ have subscripts greater than n. Therefore $p \subset L_n$ for every finite n. Hence $p \subset \prod_0^\infty L_i$, which is a contradiction. Furthermore there exists an integer $N \geq M$ so that for all $n > N$ and all i, $V_i^n \subset W_r$ for some r. For, if not, then no matter what N we take, there exists some $n > N$ and some i so the $V_i^n \not\subset W_r$, for any r. Take a sequence $\{N_j\}$ of integers such that $M < N_1 < N_2 < N_3 < \cdots$, and take the corresponding $V_{i_j}^{n_j} \not\subset W_r$, for any r. Now no $V_{i_j}^{n_j}$ will intersect both W_s and W_r, for $r, s = 1, 2, \cdots$, or h, unless the corresponding set $h_{i_j}^{n_j}$ intersects both W_s and W_r, since $W_i \cdot W_j = 0$, for $i \neq j$, and since each $V_{i_j}^{n_j} \subset W$. Hence there exists some u and some v so that each set of some subcollection $\{h_k\}$ of the collection $\{h_{i_j}^{n_j}\}$ have points in both W_u and W_v. Take a point p_k in $h_k \cdot W_u$ and a point q_k in $h_k \cdot W_v$, for each k. Now $\{p_k\}$ contains a convergent subsequence $\{p'_k\}$ which converges to a point $p \subset \prod_0^\infty L_i$, and $\{q_k\}$ contains a convergent subsequence $\{q'_k\}$ which converges to a point $q \subset \prod_0^\infty L_i$. Obviously p and q are distinct points. Furthermore, since p'_k and q'_k are on the same set $h_k = T^{-1}(b_k)$, for some $b_k \in B$, and since T is continuous, it follows that $p + q \subset T^{-1}(b)$, for some $b \in B$. Hence this set $T^{-1}(b)$ is a set of G_ω, which contradicts our assumption that $G_\omega = 0$.

The integer N depends upon ϵ. Take any $x \in A$. Now

$$Z_n(x) = S_n^{-1} T_n \cdots S_{N+1}^{-1} T_{N+1} Z_N(x)$$

and

$$Z_{n+p}(x) = S_{n+p}^{-1} T_{n+p} \cdot \cdot \cdot S_n^{-1} T_n \cdot \cdot \cdot S_{N+1}^{-1} T_{N+1} Z_N(x).$$

If $Z_N(x) \mathfrak{C} \sum_{j=N+1}^{\infty} \sum_{i=1}^{\infty} V_i^j$, then $S_k^{-1} T_k Z_N(x) = Z_N(x)$, for all $k > N$. Therefore $Z_n(x) = Z_N(x)$. Likewise $Z_{n+p}(x) = Z_N(x)$, for $p = 1, 2, \cdot \cdot \cdot$. Hence $\rho(Z_n(x), Z_{n+p}(x)) = 0$, for $n > N$ and $p = 1, 2, \cdot \cdot \cdot$, in this case. If $Z_N(x) \subset \sum_{j=N+1}^{\infty} \sum_{i=1}^{\infty} V_i^j$, then $Z_N(x) \subset W_r$ for some $r = 1, 2, \cdot \cdot \cdot$, or h. If $Z_N(x) \subset V_{d_1}^{N+1}$, for some d_1, then $S_{N+1}^{-1} T_{N+1} Z_N(x)$ is a point of $V_{m_1}^{N+1}$, for some m_1, where $V_{m_1}^{N+1} \subset W_r$. For, by Corollary II, there is a chain of neighborhoods $V_{d_1}^{N+1}, V_{e_1}^{N+1}, \cdot \cdot \cdot, V_{m_1}^{N+1}$, so that each two consecutive neighborhoods in the chain intersect. Hence $V_{m_1}^{N+1} \subset W_r$ since none of the V^{N+1}'s lying in any of the other W_i's will intersect those in W_r, as $W_i \cdot W_j = 0$, for $i \neq j$. Therefore $S_{N+1}^{-1} T_{N+1} Z_N(x) \subset W_r$ for this case. If $Z_N(x) \mathfrak{C} \sum_{i=1}^{\infty} V_i^{N+1}$, then we have that $S_{N+1}^{-1} T_{N+1} Z_N(x) = Z_N(x) \subset W_r$. If $S_{N+1}^{-1} T_{N+1} Z_N(x) \subset V_{d_2}^{N+2}$, for some d_2, then $S_{N+2}^{-1} T_{N+2} S_{N+1}^{-1} T_{N+1} Z_N(x)$ is a point of $V_{m_2}^{N+2}$, for some m_2, where $V_{m_2}^{N+2} \subset W_r$, by the same argument as above. If $S_{N+1}^{-1} T_{N+1} Z_N(x) \mathfrak{C} \sum_{i=1}^{\infty} V_i^{N+2}$, then we have that $S_{N+2}^{-1} T_{N+2} S_{N+1}^{-1} T_{N+1} Z_N(x) = S_{N+1}^{-1} T_{N+1} Z_N(x) \subset W_r$. Continue in this same way. Finally if the set $S_{n-1}^{-1} T_{n-1} \cdot \cdot \cdot S_{N+1}^{-1} T_{N+1} Z_N(x) \subset V^n{}_{d_n}$, for some d_n, then $S_n^{-1} T_n S_{n-1}^{-1} T_{n-1} \cdot \cdot \cdot Z_N(x) \subset V^n{}_{m_n}$, for some m_n, where $V^n{}_{m_n} \subset W_r$. If

$$S_{n-1}^{-1} T_{n-1} \cdot \cdot \cdot S_{N+1}^{-1} T_{N+1} Z_N(x) \mathfrak{C} \sum_{i=1}^{\infty} V_i^n$$

then

$$S_n^{-1} T_n S_{n-1}^{-1} T_{n-1} \cdot \cdot \cdot Z_N(x) = S_{n-1}^{-1} T_{n-1} \cdot \cdot \cdot Z_N(x) \subset W_r.$$

Therefore $Z_n(x)$ is a point p of W_r, and likewise

$$Z_{n+p}(x) = S_{n+p}^{-1} T_{n+p} \cdot \cdot \cdot S_{n+1}^{-1} T_{n+1}(p) = q \, \varepsilon \, W_r.$$

Accordingly

$$\rho(Z_n(x), Z_{n+p}(x)) = \rho(p, q) < \epsilon, \text{ for } n > N, \; p = 1, 2, \cdot \cdot \cdot;$$

and so $\{Z_n\}$ converges uniformly to Z. Hence Z is continuous.

Now Z transforms any set $g \, \varepsilon \, G$ into a single point of A. Furthermore, for any point $a \, \varepsilon \, A$, $Z^{-1}(a)$ is either a point p of A or a set $g \, \varepsilon \, G$ in A. Consider $T Z^{-1}(A) = B$. If $Z^{-1}(a) = p \, \varepsilon \, A$, $T Z^{-1}(a) = T(p) \, \varepsilon \, B$. If $Z^{-1}(a) = g$, where $g = T^{-1}(b)$, for some $b \, \varepsilon \, B$, then $T Z^{-1}(a) = T(g)$ which

is the point b. Hence $T Z^{-1}(A) = B$ is univalued. It is obviously continuous, since T and Z are continuous and A is compact.

Now $[T Z^{-1}]^{-1}(B) = Z T^{-1}(B) = A$ is also univalued. For take any point $b \in B$. If $T^{-1}(b) = p \in A$, then $Z T^{-1}(b) = Z(p)$ which is a point of A, since Z is univalued. If $T^{-1}(b) = g \in G$ in A, then $Z T^{-1}(b) = Z(g)$ which is also a point of A by the definition of Z.

Therefore $T Z^{-1}(A) = B$ is continuous and is univalued both ways. Hence, since A is compact, it is a homeomorphism.

8. *Applications.* The following two theorems classify some of the types of sets into which we can decompose the compact Euclidean 3-space,[12] so that, if the decomposition is upper semi-continuous and conditions (2) and (3) of the last theorem are satisfied, the hyperspace of the decomposition is also a compact Euclidean 3-space.

THEOREM a. *Let A be a compact Euclidean 3-space and $T(A) = B$ a continuous transformation such that each non-degenerate set $T^{-1}(b)$, for $b \in B$, is an isotopic sphere*[13] *plus its interior. Define the collections G and G_i and the sets L_i, for each i, as before. If there exists a number α of the first or second number class so that $G_\alpha = 0$; and if $\prod_0^\infty L_i$ is a zero-dimensional set; then B is also a compact Euclidean 3-space.*

Proof. Take any $\epsilon > 0$, any set $g \in G$, any $x \in g$, and any geometric sphere C in A. Let $S = C +$ its interior. There exists a homeomorphism $Z(A) = A$ so that $Z(S) = g$, since g is an isotopic sphere plus its interior. Let $Z^{-1}(x) = y$. Now $y \subset S$. It is well known that there exists a homeomorphism $R(A - y) = A - S$ which is stationary outside of the δ_ϵ-neighborhood U_δ of S. Let $W(A - x) = Z R Z^{-1}(A - x) = A - g$. Now W is a homeomorphism which is stationary outside of $V_\epsilon(g)$ by the proof used in the lemma for similar transformations. Therefore the conditions of the theorem in the last section are satisfied. Accordingly A is homeomorphic with B.

THEOREM b. *Let A be a compact Euclidean 3-space and $T(A) = B$ a continuous transformation such that each non-degenerate $T^{-1}(b)$, for $b \in B$, is a convex set in A. Define the collections G and G_i, and the sets L_i, for each i, as before. If there exists a number α of the first or second number*

[12] That is, the ordinary Euclidean 3-space compactified by adding the points at infinity.

[13] A set S in A is said to be an *isotopic sphere* if, for any geometric sphere C in A there exists a homeomorphism $R(A) = A$ such that $R(C) = S$.

class so that $G_a = 0$; *and if* $\prod_0^\infty L_i$ *is a zero-dimensional set; then* B *is also a compact Euclidean 3-space.*

Proof. We again take any $\epsilon > 0$, any set $g \in G$, and any $x \in g$ in A. Also take a convex ϵ-neighborhood V_ϵ of g in A. Take a system of spherical coördinates in A with the point x as origin. Choose any θ and any ϕ. The ray determined by this choice will intersect $F(V_\epsilon)$ in a single point, say (d, θ, ϕ), and it will intersect $F(g)$ in a single point, say (c, θ, ϕ). On this ray set up the following homeomorphism: $r' = \frac{(d - r) \cdot c}{d} + r$, if $r \leq d$, and $r' = r$, if $r > d$, where (r, θ, ϕ) represents any point on this ray. If we let θ and ϕ vary over all possible values, we get a homeomorphism $W(A - x) = A - g$ which is stationary outside of V_ϵ. Therefore, by the theorem of section 7, A is homeomorphic with B.

Since conditions I and III of section 2 were shown in section 3 to be equivalent when A is a 2-dimensional manifold, and in view of the theorem of section 7, we have the following:

THEOREM C. *The hyperspace* B *of any upper semi-continuous decomposition* G *of a 2-dimensional manifold* A, *where* G *is such that, for any* $g \in G$ *and any* $x \in A$, *there exists a homeomorphism* $W(A - x) = A - g$, *and such that conditions* (2) *and* (3) *of the theorem of section 7 are satisfied, is the same kind of 2-dimensional manifold, i. e. is homeomorphic with* A.

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VITA.

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